THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Screening Test – Ramanujan Contest

NMTC at INTER LEVEL – XI & XII Standards

Saturday, 7th October, 2023

Note:

1. Fill in the response sheet with your Name, Class and the name of
2. Diagrams are only visual aids; they are NOT drawn to scale.
3. You are free to do rough work on separate sheets.
4. Duration of the test: 2 hours.

PART – A

1. A function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, satisfies $f(f(x)) = f(x+2)-3$ for all real numbers x. If $f(1) = 4$ and $f(4) = 3$, the value of $f(5)$ is

   A. 6 B. 9 C. 12 D. 15

2. The numbers from 1 to 25 are each written on separate slips of paper which are placed in a pile. You draw slips from the pile without replacing any slip you have chosen. You can continue drawing until the product of two numbers on any pair of slips you have chosen is a perfect square. The maximum number of slips you can choose before you will be forced to quit is

   A. 13 B. 14 C. 15 D. 16

3. A gold number is a positive integer which can be expressed in the form $ab+a+b$, where a and b are positive integers. The number of gold numbers between 1 and 100 inclusive is

   A. 69 B. 72 C. 75 D. 78

4. The perimeter of a square lawn consists of four straight paths. Ria and Lila started at the same corner at the same time, running clockwise at constant speeds of 12 and 10 kilometres per hour respectively. Ria finished one lap around the lawn in 1 minute. During this minute, the number of seconds that Ria and Lila were on the same path is

   A. 36 B. 42 C. 48 D. 50

5. A doctor's office has a row of chairs, two of which are already occupied. Saket and Samrud come in and want to sit somewhere in the row, in adjacent chairs, that is, beside each other without a gap. Because of social distancing, there must be at least two empty chairs between them and the persons already sitting. How many chairs at least must the row contain so that Saket and Samrud can always find a place to sit together, regardless of the location of the occupied chairs?

   A. 11 B. 14 C. 15 D. 17

6. A right triangle has an angle of 60° and the hypotenuse has length 2 cm. An ant walks around the outside of the triangle always maintaining a distance of exactly 1 cm to the nearest point on the triangle, and ending where it started. The distance (in cm) travelled by the ant is

   A. $3+\sqrt{3}+\pi$ B. $3+\sqrt{3}+3\pi/2$ C. $3+\sqrt{3}+2\pi$ D. $3+\sqrt{3}+5\pi/2$

7. Consider the polynomials $P(x)$ of degree 5 with leading coefficient 1 and such that $P(-x) = -P(x)$ for all real x. If -2 and $\sqrt{3}$ are two of the roots of $P(x)$, then $P(3)$ equals

   A. 90 B. 60 C. 30 D. Can not be determined

8. In triangle ABC, D is a point on BC such that $\angle BAD = 30°$ and $\angle DAC = 15°$. If $AB = 3\sqrt{2}$ and $AC = 6$, the length of AD is

A. $2\sqrt{6}$ B. 5 C. $3\sqrt{3}$ D. $7/\sqrt{2}$

9. Three candles which can burn for 30, 40 and 50 minutes respectively are lit at different times. All three candles are burning simultaneously for 10 minutes, and there is a total of 20 minutes in which exactly one of them is burning. The number of minutes in which exactly two of them are burning is

A. 35 B. 45 C. 70 D. 90

10. All positive integers which can be expressed as a sum of one or more different integer powers of 5 are written in increasing order. The first three terms of this sequence are 1, 5 and 6. The fiftieth term is

A. 3751 B. 3755 C. 3756 D. 3760

11. We wish to choose some of the positive integers from 1 to 1000 inclusive, such that no two differ by 3 or 5. The maximum number of positive integers we can choose is

A. 200 B. 300 C. 400 D. 500

12. A country has 100 cities numbered from 1 to 100. For all $m < n$, there is a road linking the m-th and the n-th city if and only if $\frac{n}{m}$ is a prime number. For example, the city 1 is connected to all prime number cities and city 2 is connected to cities $6, 10, 14, \ldots$. One can travel in either direction on a road. The minimum number of roads one needs to use to go from the 99th city to the 100th city is

A. 8 B. 7 C. 6 D. 5

13. Arya writes down 28 consecutive numbers. If both the smallest and the largest number are perfect squares, what is the smallest number she writes down?

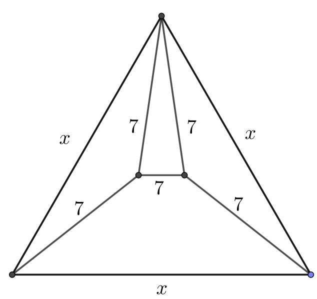
A. 9 B. 100 C. 169 D. Not uniquely determined

14. An integer with 2023 digits has the property that every two consecutive digits form a number that is a product of four prime numbers (not necessarily distinct). The digit in the 1000th position is

A. 4 B. 5 C. 6 D. 8

15. In the figure below, numbers indicate the lengths of the corresponding segments. What is the value of x?

A. $\sqrt{19}$ B. 13 C. $\sqrt{7}+\sqrt{13}$ D. 19



PART – B

16. ABC is an equilateral triangle and P is a point on the side BC such that $PB = 50$ and $PC = 30$. Find PA. ———

17. A billiard table is placed in the coordinate plane such that its bottom left corner is at the origin $(0, 0)$ and the top right corner is at $(44, 88)$ and the walls aligned with the coordinate axes. A ball placed at the point $(22, 55)$ is hit such that it meets the right wall at the point $(44, 77)$. Assume that the ball is hit without spin, the table is smooth and all the bounces are such that the angle of incidence equals the angle of reflection. If the ball hits a wall for the sixth time (including the first hit at the right most wall) at the point (a, b), then $a + b$ is ———.

18. Let x be a real number satisfying $\dfrac{(1+x)^2}{1+x^2} = \dfrac{20}{23}$. If $\dfrac{(1+x)^3}{1+x^3} = \dfrac{m}{n}$, where m, n are positive coprime integers, the value of $m + n$ is ———.

19. Let ABC be a triangle with side lengths $5, 4\sqrt{2},$ and 7. If the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$ can be expressed in the form $\dfrac{m}{n}$, where m, n are positive coprime integers, find $m + n$. ———

20. For a family gathering, 8 people order one dish each. The family sits around a circular table. Find the number of ways to place the dishes so that each person's dish is either to the left, right, or directly in front of them. ———

21. An isosceles right angled triangle ABC with right angle at C contains a point P in its interior such that $PA = 3, PB = 5$ and $PC = 2\sqrt{2}$. Find AC^2. ———

22. Suppose x is a real number such that $x^2 = 12x + 5$. There is a unique ordered pair of integers (m, n) such that $x^3 = mx + n$. Find $m + n$. ———

23. Call a three digit number $n = 100a + 10b + c$, where a, b, c are digits with $a \neq 0$, *amazing* if it satisfies the condition $n = (a + b + c) \times a \times b \times c$. It is known that there are only two three digit amazing numbers and that they differ by 9. Find the sum of those two amazing numbers. ———

24. Let $ABCD$ be a trapezium with AD parallel to BC, $AB = BC = CD = 5$ and $AD = 11$. If H is the is the orthocenter of triangle ACD, find the area of the quadrilateral $ABHD$. ———

25. In how many ways can one obtain 1232 by removing all but 4 digits from 1212323212123232? ———

26. Find the number of non congruent triangles whose side lengths are positive integers and whose perimeter is 23. ———

27. Let A, B, C be non negative integers such that $A+B+C = 23$. Find the largest possible value of $A \cdot B \cdot C + A \cdot C + C \cdot B + B \cdot A$, where for two numbers x, y, $x \cdot y$ denotes their product. ———

28. Let x, y be real numbers such that $|x| \neq |y|$ and

$$x^3 = 13x + 3y$$
$$y^3 = 3x + 13y$$

Find the value of $(x^2 - y^2)^2$. ———

29. A triple $(a, a+1, a+2)$ of integers is such that a is a multiple of 5, $a+1$ is a multiple of 7 and $a + 2$ is a multiple of 9. Find a such that it is the smallest integer greater than 1000. ———

30. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that if $i < j$, then $f(i) \leq f(j)$ and $f(f(k)) = 3k$ for all k. Find $f(9)$. ———